Last time : from simple groups, we can

those without proper normal subgroups

construct many other groups via composition series

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = G$$

each G_i/G_{i-1} is simple

Theorem (Jordan-Hölder): the multiset of composition factors $\{G_1/G_0, G_2/G_1, \ldots, G_k/G_{k-1}\}$ of some G is unique up to isomorphism

Today : from abelian groups

we will similarly construct solvable groups

Def : A group G is called solvable if $\exists$

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = G$$

s.t. G_i/G_{i-1} is abelian, $\forall i \in \{1, \ldots, k\}$

s.t. G_i/G_{i-1}

Ex: D_{2n} is solvable

$\triangledown$

$\mathbb{Z}/n\mathbb{Z}$ the subgroup of rotations

$1 \triangleleft \mathbb{Z}/n\mathbb{Z} \triangleleft D_{2n}$

$D_{2n}/\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$

Ex: any abelian group is solvable ($1 \triangleleft G$ ✓)

(non-example) Prop: S_n for $n \geq 5$ is NOT solvable

Proof: assume S_n were solvable $\Rightarrow$ can pick a maximal subnormal series $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = S_n$ s.t. G_i/G_{i-1} abelian

can't extend the series by sticking H between G_{i-1} and G_i

Claim: $G_i/G_{i-1} \cong \mathbb{Z}/\text{prime}\cdot\mathbb{Z}$

If not, $\exists$ proper $1 \triangleleft \bar{H} \triangleleft G_i/G_{i-1}$

$\Updownarrow$ correspondence theorem

$G_{i-1} \triangleleft H \triangleleft G_i$

contradiction, because H/G_{i-1} and G_i/H are abelian, due to their being a subgroup and a quotient of the abelian G_i/G_{i-1}

$\Downarrow$

G_i/G_{i-1} simple $\Rightarrow$ (the series above) is a composition series

but S_n also has the composition series $\boxed{1 \triangleleft A_n \triangleleft S_n}$

Jordan-Hölder says that the $\equiv$ $\Longrightarrow A_n$ is abelian

Why study solvable groups? Because of Galois theory

polynomial equation of deg n $\rightsquigarrow$ a subgroup of S_n

the equation can be solved by radical $\iff$ the subgroup is solvable as above

$\exists$ "many" equations whose associated group is S_n, which isn't solvable

<u>Prop 1</u>: any subgroup of a solvable group is solvable

<u>Prop 2</u>: any quotient of a solvable group is solvable

<u>Prop 3</u>: if K and L are solvable and $\exists$ s.e.s.

$$1 \longrightarrow K \longrightarrow G \longrightarrow L \longrightarrow 1$$

then G is solvable.

Propositions 2 and 3 are proved exactly like last week.

Proof of Proposition 1: assume $\underset{\underset{H}{\vee}}{G}$ is solvable

$\Rightarrow \quad 1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = G \quad$ s.t. G_i/G_{i-1} is abelian

Take $H_i = G_i \cap H$; claim that $H_{i-1} \trianglelefteq H_i$

proof. $g \in H_i \leq G_i$, $h \in H_{i-1} \leq G_{i-1}$, $ghg^{-1} \in \begin{cases} G_{i-1} & \text{b/c } G_{i-1} \triangleleft G_i \\ H & \text{b/c } H \text{ is a subgp} \end{cases}$ $\Rightarrow ghg^{-1} \in G_{i-1} \cap H = H_{i-1}$

$\Rightarrow$ subnormal series $1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{k-1} \trianglelefteq H_k = H$

$$\begin{array}{ccc} H_i & \hookrightarrow & G_i \\ \triangledown\!/ & & \triangledown\!/ \\ H_{i-1} & \hookrightarrow & G_{i-1} \end{array} \quad \rightsquigarrow \quad \text{induced } H_i/H_{i-1} \hookrightarrow G_i/G_{i-1}$$

$$\begin{array}{ccc} [h] & \rightsquigarrow & [h] \\ \| & & \|? \\ [h'] & \rightsquigarrow & [h'] \end{array}$$

? holds because $h'h^{-1} \in H_{i-1} < G_{i-1}$

because subgroups of abelian groups are abelian, conclude that H_i/H_{i-1} is abelian

$1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_{k-1} \trianglelefteq H_k = H$ implies H is solvable (after remove redundancies)

$1 \triangleleft \mathbb{Z}/n\mathbb{Z} \triangleleft D_{2n} \quad \Leftarrow \quad 1 \triangleleft \cancel{\mathbb{Z}/n\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z} = \cancel{\mathbb{Z}/n\mathbb{Z}} \triangleleft D_n$

New characterization of solvable groups:

<u>Def</u>: given a group G and a set $Z \subseteq G$, the subgroup generated by Z is $H = \left\{ z_1^{\pm 1} z_2^{\pm 1} \cdots z_k^{\pm 1}, \ \forall z_1, \dots, z_k \in Z \right\} \leq G$

<u>Def</u>: given a group G and two subsets $X, Y \subseteq G$, their subgroup of commutators $[X, Y] \leq G$ is the subgroup generated by $\{ xyx^{-1}y^{-1} \mid x \in X, y \in Y \}$

note that $(xyx^{-1}y^{-1})^{-1} = yxy^{-1}x^{-1}$, so

$$[X,Y] \ni (x_1 y_1 x_1^{-1} y_1^{-1})(x_2 y_2 x_2^{-1} y_2^{-1})(y_3 x_3 y_3^{-1} x_3^{-1}) \dots (x_k y_k x_k^{-1} y_k^{-1})$$

<u>Def</u>: for a group G, its derived subgroup is

$$[G, G] \leq G$$

(note that $1 = [G, G] \iff G$ is abelian)

Prop: $\forall$ group G • we have $[G,G]$ is normal in G

• $G/[G,G]$ is abelian

• this is the biggest abelian quotient of G, i.e

if $H \trianglelefteq G$ is such that G/H is abelian, then $[G,G] \leq H$

Proof • $\forall$ $x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \dots\dots x_k y_k x_k^{-1} y_k^{-1} \in [G,G]$

$\forall$ $g \in G$, we have

$$g\left(x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1}\, y_3 x_3 y_3^{-1} x_3^{-1} \dots\dots x_k y_k x_k^{-1} y_k^{-1}\right) g^{-1}$$

$$\|$$

$$(g x_1 g^{-1})(g y_1 g^{-1})(g x_1 g^{-1})^{-1}(g y_1 g^{-1})^{-1} \dots\dots (g x_k g^{-1})(g y_k g^{-1})(g x_k g^{-1})^{-1}(g y_k g^{-1})^{-1}$$

Since this lies in $[G,G]$, we conclude that $[G,G]$ is normal in G

• $[x],[y] \in G/[G,G]$, $[x][y][x]^{-1}[y]^{-1} = [x y x^{-1} y^{-1}] = e \in G/[G,G]$

$$\Downarrow$$

$[x]\cdot[y] = [y]\cdot[x]$ in $G/[G,G]$

• $H \trianglelefteq G$ s.t. G/H is abelian

$\forall$ $x,y \in G$, $[x],[y] \in G/H$ $\Bigg\}$ $\Rightarrow$ $[x][y] = [y][x] \in G/H$

$\Downarrow$ $\Downarrow$

$[x][y][x]^{-1}[y]^{-1} = e$

$$\Downarrow$$

$$[xyx^{-1}y^{-1}] = e$$

$$\Downarrow$$

$$[G,G] \leq H \Longleftarrow xyx^{-1}y^{-1} \in H$$

<u>Def</u>: $\forall$ group G, its derived series is

$$\cdots\cdots \trianglelefteq G^{(2)} \trianglelefteq G^{(1)} \trianglelefteq G^{(0)} = G$$

where $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$, $\forall\, i \geq 1$

<u>Prop</u>: the derived series terminates $\rightsquigarrow G^{(k)} = 1$ for some k

$$\Updownarrow$$

G is solvable

<u>Proof</u>: $\Downarrow$ is obvious from the definition of solvable

$\Uparrow$ $\exists$ series $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = G$ s.t G_i/G_{i-1} is abelian $\forall i$

but given that G_k/G_{k-1} is abelian $\xRightarrow{\text{Prop}} G_{k-1} \geq \overset{\overset{G^{(0)}}{\shortparallel}}{[G_k, G_k]} = G^{(1)}$

but given that G_{k-1}/G_{k-2} is abelian $\Longrightarrow G_{k-2} \geqslant [G_{k-1}, G_{k-1}] \geqslant [G', G'] = G''$

⋮

$1 = G_0 \geqslant G^{(k)}$ $\Rightarrow$ derived series terminates

<u>Def</u>: A group G is nilpotent if its lower central series

$$\cdots\cdots \trianglelefteq G^{\{2\}} \trianglelefteq G^{\{1\}} \trianglelefteq G^{\{0\}} = G$$

(where $G^{\{i\}} = [G^{\{i-1\}}, G]$, $\forall i \geq 1$) eventually terminates

i.e. $G^{\{k\}} = 1$ for some k

- $G^{\{i\}} \geqslant G^{(i)}$ $\Rightarrow$ any nilpotent is solvable
- any abelian group is nilpotent, because $G^{\{1\}} = 1$
- in general, $G^{\{k\}} = 1$ iff all k-fold commutators are the identity

0-fold commutator is $g_0 \in G$

1-fold commutator is $g_0 g_1 g_0^{-1} g_1^{-1}$, $\forall g_0, g_1 \in G$

2-fold commutator is $(g_0 g_1 g_0^{-1} g_1^{-1}) g_2 (g_1 g_0 g_1^{-1} g_0^{-1}) g_2^{-1}$, $\forall g_0, g_1, g_2 \in G$

⋮

Prop: lower central series is actually a normal series, i.e

$$G^{\{i\}} \trianglelefteq G$$

Proof: by induction on i (the base case $i=0$ is trivial)

for the induction step, assume $G^{\{i-1\}}$ is normal in G

$$G^{\{i\}} = [G^{\{i-1\}}, G] = \left\{ \text{products } (xyx^{-1}y^{-1})^{\pm 1} \;\middle|\; x \in G^{\{i-1\}}, y \in G \right\}, \text{ so}$$

$$g \cdot \left(\text{product of } (xyx^{-1}y^{-1})^{\pm 1}\right) \cdot g^{-1} = \text{product of} \Big[\underset{\substack{\cap \\ G^{\{i-1\}} \\ \text{by ind hyp}}}{(gxg^{-1})} \underset{\substack{\cap \\ G}}{(gyg^{-1})} \underset{\substack{\cap \\ G^{\{i-1\}}}}{(gxg^{-1})^{-1}} \underset{\substack{\cap \\ G}}{(gyg^{-1})^{-1}} \Big]^{\pm 1}$$

$$\cap$$

$$G^{\{i\}}, \ \forall g \in G$$

$$\Downarrow$$

$G^{\{i\}}$ is normal in G $\square$

example of this computation

$$g(h_1 h_2^{-1} h_3 h_4^{-1}) g^{-1} = (g h_1 g^{-1})(g h_2 g^{-1})^{-1}(g h_3 g^{-1})(g h_4 g^{-1})^{-1}$$

$$\| \qquad\qquad \|$$

$$g h_1 \underbrace{g^{-1} g}_{e} h_2^{-1} \underbrace{g^{-1} g}_{e} h_3 \underbrace{g^{-1} g}_{e} h_4^{-1} g$$

In Lie theory, one encounters the following examples of abelian, nilpotent, solvable (

square matrices, with coefficients in any field (even infinite)